Lecture 06: Expected Max-Load & Poisson Approximation Theorem

Experiment

- There are $m = n$ balls
- Each ball is thrown uniformly and independently at random into n bins
- **Objective.** Understand the behavior of $\mathbb{E}\left[\mathbb{L}_{\mathsf{max}}\right]$

Bounding the expected value of $\mathbb{L}_{\text{max}}$

We shall show the following result

Theorem (Expected Max-Load)

Let $m = n$ balls be thrown uniformly and independently at random into n bins. Let $\mathbb{L}_{\text{max}}$ be the random variable denoting the maximum load of the bins. Then, we have the following result.

$$\mathbb{E}\left[\mathbb{L}_{\text{max}}\right] = \Theta\left(\frac{\log n}{\log\log n}\right)$$

Upper Bound I

- Our idea is to prove the following. For some positive constant c, we have
$$\mathbb{E}\left[\mathbb{L}_{\mathsf{max}}\right] \leqslant c\left(\frac{\log n}{\log\log n}\right)$$
- Our strategy is to use the following trick to calculate the expectation of a random variable $\mathbb{X}$ over natural numbers

$$\begin{aligned}\mathbb{E}\left[\mathbb{X}\right] &= \sum_{i \geqslant 1} i \cdot \mathbb{P}\left[\mathbb{X} = i\right] \\ &= \sum_{i \geqslant 1} \sum_{j \geqslant i} \mathbb{P}\left[\mathbb{X} = j\right] \\ &= \sum_{i \geqslant 1} \mathbb{P}\left[\mathbb{X} \geqslant i\right]\end{aligned}$$

Upper Bound II

- So, we have

$$\mathbb{E}\left[\mathbb{L}_{\max}\right] = \sum_{i \geqslant 1} \mathbb{P}\left[\mathbb{L}_{\max} \geqslant i\right]$$

- We begin with the following result

Lemma

For any $\ell \in \mathbb{N}$, we have the following bound

$$\mathbb{P}\left[\mathbb{L}_j \geqslant \ell\right] \leqslant \binom{n}{\ell} \frac{1}{n^\ell} \leqslant \frac{1}{\ell!}$$

Proof Outline.

- The probability that bin j receives $\geqslant \ell$ balls is (at most) the probability of the following event
  - We choose a set of ℓ balls from n balls in $\binom{n}{\ell}$ ways
  - We compute the probability that these ℓ balls land in bin j
  - The other balls can go anywhere (including falling in bin j)

Upper Bound III

Food for Thought. Why is this probability expression an inequality and not an equality?

- Let ℓ^* be the smallest integer such that $(\ell^*)! \geqslant n^2$
- **Exercise.** Prove that $\ell^* \leqslant c\frac{\log n}{\log\log n}$ for some positive constant c
- So, we have $\mathbb{P}\left[\mathbb{L}_j \geqslant \ell^*\right] \leqslant 1/n^2$
- Now, by Union Bound, we have

$$\mathbb{P}\left[\mathbb{L}_1 \geqslant \ell^* \text{ or } \mathbb{L}_2 \geqslant \ell^* \text{ or } \cdots \text{ or } \mathbb{L}_n \geqslant \ell^*\right] \leqslant n \cdot \frac{1}{n^2} = \frac{1}{n}$$

- That is, we have

$$\mathbb{P}\left[\mathbb{L}_{\max} \geqslant \ell^*\right] \leqslant \frac{1}{n}$$

- Now, we are at a position to upper-bound the expected max-load

$$\begin{aligned}\mathbb{E}\left[\mathbb{L}_{\max}\right] &= \sum_{i \geqslant 1} \mathbb{P}\left[\mathbb{L}_{\max} \geqslant i\right] \\ &= \sum_{i=1}^{\ell^*-1} \mathbb{P}\left[\mathbb{L}_{\max} \geqslant i\right] + \sum_{i=\ell^*}^{n} \mathbb{P}\left[\mathbb{L}_{\max} \geqslant i\right] \\ &= (\ell^* - 1) \cdot 1 + (n - \ell^*) \cdot \frac{1}{n} \\ &< \ell^*\end{aligned}$$